EE278 Statistical Signal Processing **Stanford, Autumn 2023**

Homework 8

Due: Thursday, December 7, 2023, 1:00 pm on Gradescope

Please upload your answers timely to Gradescope. Start a new page for every problem. For the programming/simulation questions you can use any reasonable programming language. Comment your source code and include the code and a brief overall explanation with your answers.

1. (**10 pts**) Consider the n-dimensional multi-sensor estimation problem:

$$\mathbf{Y} = \mathbf{h}X + \mathbf{W}, \qquad X \sim \mathcal{N}(0, \sigma_x^2), W \sim \mathcal{N}(0, \sigma_w^2 \mathbf{I}),$$

and X and $\mathbf{W}$ are independent.

   a) (**3 pts**) Derive the MMSE estimator of X given $\mathbf{Y}$ using the general formula you derived for Q. 1 in HW 7 Is the relevant matrix invertible? Comment on the computational effort of directly implementing this estimator for n large.

   b) (**3 pts**) Derive an estimator of X by first projecting $\mathbf{Y}$ along $\mathbf{h}$ to obtain V and then compute the MMSE estimate of X given V. Comment on the computational effort of implementing this estimator for n large.

   c) (**4 pts**) Show that the estimators in (a) and (b) are identical. (Hint: let $\mathbf{g}^T$ be the row vector $K_{XY}K_Y^{-1}$ and multiply it by K_Y to solve for $\mathbf{g}$.)

2. (**10 pts**) Consider the dynamical system discussed in class

$$\begin{aligned} X_0 &\sim \mathcal{N}(0, \sigma_0^2) \\ X_n &= \alpha X_{n-1} + W_{n-1} \qquad n = 1, 2, \ldots \\ Y_n &= X_n + Z_n, \qquad n = 0, 1, 2 \ldots \end{aligned}$$

with the W_n's i.i.d. $\mathcal{N}(0, \sigma_w^2)$ random variables and Z_n's i.i.d. $\mathcal{N}(0, \sigma_z^2)$ random variables, all independent of each other and independent of X_0. We focus on the case $\alpha = 1$ and $\sigma_w^2 = 0$.

   a) (**5 pts**) Compute the MMSE estimate of X_n given $\mathbf{Y}^n = [Y_0, \ldots, Y_n]^T$ using the result of the previous problem.

   b) (**5 pts**) Check that the Kalman filter recursion gives the same answer.

3. (**12 pts**) Consider the same dynamical system as in Q. 2:

$$\begin{aligned} X_0 &\sim \mathcal{N}(0, \sigma_0^2) \\ X_n &= \alpha X_{n-1} + W_{n-1} \qquad n = 1, 2, \ldots \\ Y_n &= X_n + Z_n, \qquad n = 0, 1, 2 \ldots \end{aligned}$$

with the W_n's i.i.d. $\mathcal{N}(0, \sigma_w^2)$ random variables and Z_n's i.i.d. $\mathcal{N}(0, \sigma_z^2)$ random variables, all independent of each other and independent of X_0. In this question, we will focus on the case $0 < \alpha < 1$

a) (**6 pts**) Let σ_n^2 be the variance of X_n. Show that σ_n^2 increases or decreases monotonically to a limit σ_∞^2 as $n \to \infty$ and identify the limit. Hence, show that X_n and Y_n both converge to steady-state distributions as $n \to \infty$ and identify the steady-state distributions. (Hint: find a relationship between $\sigma_n^2 - \sigma_{n-1}^2$ and $\sigma_{n+1}^2 - \sigma_n^2$.)

b) (**6 pts**) Fix $\sigma_0^2 = \sigma_z^2 = \sigma_\infty^2 = 1$. Consider 4 possible values for α: $\alpha = 0.1, 0.5, 0.9, 0.99$. For each of these values, simulate the system and plot a realization of $\{X_n\}$ and a realization of $\{Y_n\}$ on the same plot. Explain how the plot qualitatively changes as α varies.

4. (**18 pts**) We apply the Kalman filter to generate estimates $\hat{X}_n$'s to track the dynamical system in Q. 3. Let v_n^2 be the MMSE error in estimating X_n. We will continue to assume $0 < \alpha < 1$.

a) (**3 pts**) Show that $v_n^2 < \sigma_n^2$. (Hint: no calculations are needed.)

b) (**3 pts**) Show that v_n^2 is monotonic in n. (Hint: use the same technique as in Q. 2(a).)

c) (**3 pts**) Using (a) and (b) or otherwise, show that v_n^2 converges to a limit. Compute the limit v_∞^2.

d) (**3 pts**) Compute the MMSE estimate $\tilde{X}_n$ of X_n based on Y_n *only*, and compute the resulting MMSE error. Compute the limit e_∞^2 of this error as $n \to \infty$.

e) (**3 pts**) Fix $\sigma_0^2 = \sigma_z^2 = \sigma_\infty^2 = 1$. Plot both v_∞^2 and e_∞^2 as a function of α between 0 and 1. For which value of α is the gain from using the entire past history of the observations rather than just the current observation greatest? For what value of α is the gain smallest?

f) (**3 pts**) Fix $\sigma_0^2 = \sigma_z^2 = \sigma_\infty^2 = 1$. Consider 4 possible values for α: $\alpha = 0.1, 0.5, 0.9, 0.99$. For each of these values, simulate the system and plot the resulting trajectories of $\{X_n\}$, $\{\hat{X}_n\}$ and $\{\tilde{X}_n\}$ and on the same plot. Explain how the plot qualitatively changes as α varies.